Toy model for a two-dimensional accretion disk dominated by Poynting flux

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We discuss the effect of the Poynting flux on a magnetically dominated thin accretion disk, which is simplified to a two-dimensional disk on the equatorial plane. It is shown in the relativistic formulation that the Poynting flux caused by a rotating magnetic field with Keplerian angular velocity can balance the energy and angular momentum conservation of a stationary accretion flow.

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I. INTRODUCTION

The Poynting flux model [1,2], suggested for accretion disks with ordered magnetic fields, has been considered to be one of the viable models for astrophysical jets [3,4]. In contrast with the hydrodynamic jets in which the energy and the angular momentum are carried by the kinetic flux of matter, the Poynting flux is characterized by the outflow of energy and angular momentum carried predominantly by the electromagnetic field.

In the nonrelativistic formulation, Blandford [2] suggested an axisymmetric and stationary solution for the Poynting outflow assuming a force-free magnetosphere surrounding an accretion disk. The poloidal field configuration for a black hole in a force-free magnetosphere was discussed recently by Ghosh [5] in the relativistic formulation, where possible forms of the poloidal configuration of the magnetic field are suggested. The development of the ordered magnetic field in the disk and the Poynting outflow from the disk have been studied by many authors [6–8]. Recently Ustyugova *et al.* [4] performed an axisymmetric magnetohydrodynamical simulation to show that a quasistationary and approximately force-free Poynting jet from the inner part of the accretion disk is possible.

The Poynting flux in a system of black hole and accretion disk recently has also been studied in connection with gamma ray bursts [9–11]. One of the advantages of considering the Poynting flux is that it carries a very small baryonic component, which is essential for powering the gamma ray bursts [12]. The evolution of the system is also found to be largely dependent on the Poynting outflow from the disk [13,14]. The relativistic effects on accreting flows close to a black hole have been discussed by Lasota [15], Abramowicz *et al.* [16], and recently by Gammie and Popham [17], where the relativistic effects on a slim disk with averaged vertical structures are discussed in detail, including viscous stresses. However, the effect of relativity on the accreting flow dominated by Poynting flux has not been discussed in depth so far.

The purpose of this work is to study the effect of Poynting flux on the accretion flow in the relativistic formulation. In this work, we consider a toy model for a magnetically dominated thin accretion disk, which is assumed to be a two-dimensional disk located on the equatorial plane with a black hole at the center. To see the magnetic effect transparently it is also assumed that there is no viscous stress tensor in the disk and there is no radiative transfer from the disk. We develop a relativistic description of a two-dimensional model for a Poynting-flux-dominated thin accretion disk in the background metric of a Kerr black hole. It is shown that the energy and angular momentum balances of the accretion disk for a stationary accretion flow can be maintained by Poynting flux, provided that the poloidal magnetic fields are rotating with the same Keplerian angular velocities Ω_K as the stable orbits in the equatorial plane.

II. TWO-DIMENSIONAL ACCRETION FLOW

The stress-energy tensor $T_m^{\mu\nu}$ of the matter in the disk is given by

$$T_m^{\mu\nu}=(\rho_m+p+\Pi)u^\mu u^\nu+pg^{\mu\nu}+S^{\mu\nu}+u^\mu q^\nu+u^\nu q^\mu, \tag{1}$$

where ρ_m, Π, and p are the rest-mass density, the internal energy, and the pressure, respectively. $S^{\mu\nu}$ is the viscous tensor and q^μ is the radiative energy flux [16].

In this work we simplify the accretion disk to be nonviscous ($S^{\mu\nu}=0$), cool ($p=0, \Pi=0$), and nonradiative ($q^\mu=0$) for the purpose of investigating the effect of the magnetic field transparently. It is also assumed that there is a negligible mass flow in the direction perpendicular to the disk, $u^\theta=0$. Then the stress-energy tensor of the matter is given by

$$T_m^{\mu\nu}=\rho_m u^\mu u^\nu. \tag{2}$$

We can obtain the energy flux given by

$$\mathcal{E}^\mu_{(m)}=-\rho_m u_0 u^\mu, \tag{3}$$

and the angular momentum flux by

$$\mathcal{L}^\mu_{(m)}=\rho_m u_\phi u^\mu. \tag{4}$$

For an idealized thin disk on a two-dimensional plane we take

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$$\rho_m = \frac{\sigma_m}{\rho}\,\delta(\theta - \pi/2), \tag{5}$$

where σ_m is the surface rest-mass density. In this work, the background geometry is assumed to be determined by the Kerr metric (see Appendix A) with a rotating black hole at the center, and we adopt the natural unit $G=c=1$.

The rate of rest-mass flow crossing a circle of radius r is given by

$$\int_r \alpha \rho_m u^r \frac{\rho^2 \varpi}{\sqrt{\Delta}}\, d\theta\, d\phi = 2\pi\sigma_m \rho u^r, \tag{6}$$

which defines the mass accretion rate $\dot{M}_+$ by

$$\dot{M}_+ = -2\pi\sigma_m \rho u^r. \tag{7}$$

This is identical to the one derived in [16] and [17], in which the vertical structures are averaged. The stationary accretion flow implies that $\dot{M}_+$ is r independent.

Using Eqs. (3) and (4) the radial flow of the energy of the matter at r can be given by

$$\int_r \alpha \mathcal{E}^r_{(m)} \frac{\rho^2 \varpi}{\sqrt{\Delta}}\, d\theta d\phi = -2\pi\sigma_m \rho u^r u_0 = u_0 \dot{M}_+ \tag{8}$$

and the radial flow of the angular momentum of the matter by

$$\int_r \alpha \mathcal{L}^r_{(m)} \frac{\rho^2 \varpi}{\sqrt{\Delta}}\, d\theta\, d\phi = 2\pi\sigma_m \rho u^r u_\phi = -u_\phi \dot{M}_+ . \tag{9}$$

For the magnetosphere outside the accretion disk, there are electromagnetic currents J^μ which can flow through the central object and also along the magnetic field lines anchored on the disk. In this work we suppose a situation in which the currents flow into the inner edge of the disk via the central object and then flow out along the magnetic field lines from the disk as discussed by Blandford and Znajek [2,18]. The continuity of the currents (current conservation) necessarily requires surface currents in the radial direction on the disk. A similar consideration has been applied to the black hole horizon as discussed by Thorne *et al.* [19].

In general the structure of the electromagnetic field with a discontinuity in the $\theta = \pi/2$ plane can be reproduced by assigning the surface charge and the surface current on the plane in addition to the bulk charge and bulk current distributions, which terminate on the disk. The surface charge density σ_e and the surface current density $K^{\hat{i}}$ can be defined systematically using the procedure suggested by Damour [20] (see Appendix B for details):

$$\sigma_e = -\frac{E^{\hat{\theta}}}{4\pi}, \tag{10}$$

$$K^{\hat{r}} = -\frac{1}{4\pi} B^{\hat{\phi}}, \quad K^{\hat{\phi}} = \frac{1}{4\pi} B^{\hat{r}}. \tag{11}$$

III. POYNTING FLUX

The energy and the angular momentum of the disk can be carried out by the Poynting flux along the magnetic field lines that are anchored on the disk. In our simplified model, this is the main driving force for the accretion flow. We calculate the Poynting flux on a two-dimensional disk assuming a force-free magnetosphere around the accretion disk.

Using the Killing vector in the t direction,

$$\xi^\mu = (1,0,0,0), \tag{12}$$

we can define the energy flux $\mathcal{E}^\mu$ from the energy momentum tensor $T^{\mu\nu}$,

$$\mathcal{E}^\mu = -T^{\mu\nu}\xi_\nu = (\alpha^2 - \varpi^2\beta) T^{\mu 0} - \varpi^2 \beta T^{\mu\phi}, \tag{13}$$

$$\mathcal{E}^\mu{}_{;\mu} = 0, \tag{14}$$

where

$$T^{\mu\nu} = \frac{1}{4\pi}\left(F^\mu_\rho F^{\nu\rho} - \frac{1}{4} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right). \tag{15}$$

Since we are interested in the θ direction (normal to the disk on the equatorial plane), $\mathcal{E}^\theta$ is of interest to us, for which the second term in Eq. (15) vanishes:

$$\mathcal{E}^\theta = \frac{1}{4\pi}(\alpha^2 - \varpi^2\beta) F^\theta_\rho F^{0\rho} - \frac{1}{4\pi}\varpi^2 \beta F^\theta_\rho F^{\phi\rho}. \tag{16}$$

Using the identities given in Appendix A, it can be rewritten in terms of the electric and magnetic fields as given by

$$\mathcal{E}^\theta = \frac{1}{4\pi\rho}[\alpha \hat{E} \times \hat{B}|_\theta + \beta\varpi (E^{\hat{\theta}} E^{\hat{\phi}} + B^{\hat{\phi}} B^{\hat{\theta}})]. \tag{17}$$

The power measured at infinity can be obtained using Eq. (17) by

$$P^\theta_{energy} = \int_{\theta=\pi/2} \alpha \mathcal{E}^\theta\, d\Sigma_\theta = \int \frac{\alpha}{4\pi}[\alpha(\vec{E}\times\vec{B})_\theta + \varpi\beta(E^{\hat{\theta}} E^{\hat{\phi}} + B^{\hat{\phi}} B^{\hat{\theta}})] \frac{\rho\varpi}{\sqrt{\Delta}}\, dr\, d\phi. \tag{18}$$

When the tangential components of the electromagnetic field are multiplied by the laps function α as in the membrane paradigm [19], we can hide α in the integrand.

For the steady and axisymmetric case that we are interested in, $E^{\hat{\phi}} = 0$, and we get

$$\mathcal{E}^{\theta}=\frac{1}{4\pi\rho}(-\alpha E^{\hat{r}}B^{\hat{\phi}}+\beta\varpi B^{\hat{\phi}}B^{\hat{\theta}}). \tag{19}$$

The first term in the integrand in Eq. (18) can be written in terms of the surface current density. Using Eq. (B16) we get

$$\frac{1}{4\pi}(\vec{E}\times\vec{B})_{\theta}=\vec{E}\cdot\vec{K}. \tag{20}$$

For the current in the direction of the electric field tangential to the disk there might be energy dissipation into the disk surface. This is what one can expect on the black hole horizon [19]. However, for the current in the opposite direction this term corresponds to the electromotive force and that is the case for the accretion disk on the equatorial plane discussed by Blandford and Znajek [2,18] as well as in this work.

Similarly, the third term in Eq. (18) [equivalently the second term in Eq. (19)] can be written in terms of the surface current using Eq. (B16) as given by

$$\frac{1}{4\pi}\varpi\beta B^{\hat{\phi}}B^{\hat{\theta}}=\varpi\omega K^{\hat{r}}B^{\hat{\theta}}. \tag{21}$$

We can see that this can be interpreted as a magnetic braking power on a rotating body with angular velocity $\omega=-\beta$.

Using the Killing vector in the ϕ direction for the axially symmetric case,

$$\eta^{\mu}=(0,0,0,1), \tag{22}$$

we can define the angular momentum flux $\mathcal{L}^{\mu}$ from the energy momentum tensor $T^{\mu\nu}$,

$$\mathcal{L}^{\mu}=T^{\mu\nu}\eta_{\nu}=\varpi^{2}\beta T^{\mu 0}+\varpi^{2}T^{\mu\phi}, \tag{23}$$

$$\mathcal{L}^{\mu}{}_{;\mu}=0. \tag{24}$$

Then we get the flux in the θ direction (normal to the disk on the equatorial plane) $\mathcal{L}^{\theta}$,

$$\mathcal{L}^{\theta}=-\frac{\varpi}{4\pi\rho}B^{\hat{\theta}}B^{\hat{\phi}}=\frac{1}{\rho}\varpi K^{\hat{r}}B^{\hat{\theta}}, \tag{25}$$

which is nothing but a torque exerted on the surface current density K^{r}. Equation (17) can be written as

$$\mathcal{E}^{\theta}=\frac{1}{4\pi\rho}\alpha\hat{E}\times\hat{B}|_{\theta}-\beta\mathcal{L}^{\theta}. \tag{26}$$

In this work we assume that there is sufficient ambient plasma around the disk for a force-free magnetosphere [2,4]. The force-free condition for a magnetosphere with current density J^{μ} is given by

$$F_{\mu\nu}J^{\mu}=0. \tag{27}$$

The magnetic flux Ψ through a circuit encircling $\phi=0\rightarrow 2\pi$,

$$\Psi=\oint A_{\phi}\,d\phi=2\pi A_{\phi}, \tag{28}$$

defines a magnetic surface on which $A_{\phi}(r,\theta)$ is constant. Therefore the magnetic surface can be characterized by the magnetic flux Ψ contained inside it. From the force-free condition it can be shown that A_{0} is also constant along the magnetic field lines and the electric field is always perpendicular to the magnetic surface. We can define a function $\Omega_{F}(r,\theta)$,

$$dA_{0}=-\Omega_{F}\,dA_{\phi}, \tag{29}$$

which is also constant along the magnetic surface. Ω_{F} can be identified as the angular velocity of the magnetic field line on the magnetic surface [18,19].

Then we get from the force-free condition

$$\vec{E}=-\frac{\varpi}{\alpha}(\Omega_{F}+\beta)\hat{\phi}\times\vec{B}, \tag{30}$$

and the Poynting flux perpendicular to the disk is given by

$$\mathcal{E}^{\theta}=-\frac{\Omega_{F}\varpi}{4\pi\rho}B^{\hat{\theta}}B^{\hat{\phi}}=\Omega_{F}\mathcal{L}^{\theta}_{(\phi)}, \tag{31}$$

as in [18].

IV. ENERGY AND ANGULAR MOMENTUM CONSERVATION

In the idealized thin disk considered in this work, it is assumed that there are no radiative transfers or viscous interactions which otherwise take part in balancing the radial flows of the energy and the angular momentum of the accreting matter. Hence the electromagnetic field anchored on the disk is responsible for the conservations of total energy and angular momentum. Since we assume an idealized two-dimensional disk, there is no radial flow of the electromagnetic energy and angular momentum unless there is any singular structure of the electromagnetic fields on the disk.

Let us consider a circular strip at r with infinitesimally small width δr. Using Eq. (25) the angular momentum flux of the electromagnetic field in the θ direction is given by

$$\Delta L_{EM}=2\int_{r}^{r+\delta r}\alpha\mathcal{L}^{\theta}\,d\Sigma_{\theta}=-B^{\hat{\theta}}B^{\hat{\phi}}\rho\varpi\,\delta r, \tag{32}$$

where factor 2 is introduced to take account of both sides of the disk. It should be balanced by the change of the angular momentum of the matter given by

$$\Delta L_{m}=-u_{\phi}\dot{M}_{+}|_{r}^{r+\delta r}\rightarrow-\frac{du_{\phi}}{dr}\dot{M}_{+}\delta r. \tag{33}$$

Then we get

$$\dot{M}_{+}=\frac{B^{\hat{\theta}}B^{\hat{\phi}}\rho\varpi}{du_{\phi}/dr}. \tag{34}$$

In the nonrelativistic limit u_ϕ can be identified as the nonrelativistic Keplerian angular momentum $\Omega_K = \sqrt{M/r^3}$, and we get

$$\dot{M}_+ \to \frac{B^{\hat{\theta}}B^{\hat{\phi}}r^2}{r\Omega_K/2} = 2r\frac{B^{\hat{\theta}}B^{\hat{\phi}}}{\Omega_K}, \tag{35}$$

which is identical to that used in [13] and [4].

Similarly, the energy flux of the electromagnetic field in the θ direction is given by

$$\Delta E_{EM} = 2\int_r^{r+\delta r} \alpha \mathcal{E}^\theta \, d\Sigma_\theta = \Omega_F \Delta L_{EM} = -\Omega_F B^{\hat{\theta}}B^{\hat{\phi}}\rho\varpi\,\delta r, \tag{36}$$

where a factor 2 is introduced for the same reason as in the angular momentum flux. It is balanced by the change of energy of the matter given by

$$\Delta E_m = -(-u_0)\dot{M}_+|_r^{r+dr} \to \frac{du_0}{dr}\dot{M}_+\delta r. \tag{37}$$

Using Eq. (34) it can be shown that the energy balance can be realized with Ω_F given by

$$\Omega_F = -\frac{du_0/dr}{du_\phi/dr}. \tag{38}$$

It is interesting to note that this has no explicit dependence on the electromagnetic field configuration.

It is not a trivial task to find a solution of the full relativistic magnetohydrodynamic (MHD) equations satisfying Eqs. (7) and (34), which lead to

$$B^{\hat{\theta}}B^{\hat{\phi}} = -2\pi\sigma_m \rho u^r \frac{du_\phi}{dr}, \tag{39}$$

together with Eq. (38). Without solving the full MHD equation, the magnetic field and its angular velocity Ω_F can be considered to be unknown parameters as well as u_i.

As a first trial, we assume u_0 and u_ϕ to be those of the stable orbit of a test particle around a Kerr black hole [21] given by

$$-u_0 = \frac{r^2 - 2Mr + a\sqrt{Mr}}{r\sqrt{r^2 - 3Mr + 2a\sqrt{Mr}}},$$

$$u_\phi = \frac{\sqrt{Mr}(r^2 - 2a\sqrt{Mr} + a^2)}{r\sqrt{r^2 - 3Mr + 2a\sqrt{Mr}}}, \tag{40}$$

although it is natural to expect a non-negligible effect of the magnetic field on the stable orbit. The straightforward calculation, Eq. (38), shows the very interesting result that the angular velocity Ω_F of the magnetic field is the same as the Keplerian angular velocity $\Omega_K (\equiv u^\phi/u^0)$,

$$\Omega_F = \Omega_K, \tag{41}$$

where

$$\Omega_K = \sqrt{\frac{M}{r^3}}\frac{1}{1 + a\sqrt{M/r^3}}. \tag{42}$$

In fact one can derive Eq. (41) formally without using the explicit form of Eq. (40), since it can be shown that

$$u^0\frac{du_0}{dr} + u^\phi \frac{du_\phi}{dr} = 0 \tag{43}$$

holds in general for a stable circular orbit on the equatorial plane. This shows that the energy and angular momentum balances can be realized by a magnetic field rotating rigidly with the Keplerian angular velocity in this simplified two-dimensional disk model.

It is also possible to show that Eq. (4.2) in [2] can be obtained from Eq. (34) here in the nonrelativistic limit. Hence for a Newtonian limit or for the outer radius of the thin disk, the configuration of the electromagneic field suggested by Blandford [2] is consistent with our result when the magnetic field angular velocity Ω_F is given by Ω_K.

V. POWER OUT OF A POYNTING-FLUX-DOMINATED THIN ACCRETION DISK

The total power out of the disk, P_{disk}, can be calculated by integrating Eq. (36) from the innermost stable point (r_{in}) to the outermost edge of the disk with the Poynting flux (r_{out}),

$$P_{disk} = -\int_{r_{in}}^{r_{out}} (-\Omega_F B^{\hat{\theta}}B^{\hat{\phi}}\rho\varpi)dr, \tag{44}$$

where the minus sign is referring to the outward direction on the disk.

Using the accretion rate [Eq. (34)], which is independent of r for a stationary accretion flow, and the energy balance condition [Eq. (38)], we get

$$P_{disk} = \int_{r_{in}}^{r_{out}} \Omega_F \dot{M}_+ \frac{du_\phi}{dr}dr = \dot{M}_+ \int_{r_{in}}^{r_{out}} \frac{d(-u_0)}{dr}dr \tag{45}$$

$$= [-u_0(r_{out})]\dot{M}_+ - [-u_0(r_{in})]\dot{M}_+. \tag{46}$$

The first term corresponds to the energy accretion rate into the disk at the outermost edge and the second term corresponds to the energy accretion rate at the inner edge onto the central object. This is the expected result from energy conservation.

Equation (46) indicates that the power out of the disk in this toy model is not strongly dependent on the details of the electromagnetic field configuration. The power can be calculated once $\dot{M}_+$ is known at one point r on the disk. For example, suppose that the exact solution of the relativistic equation can be approximated at a large distance r_0 by the configuration suggested by Blandford [2]; then we can use the relation, for example,

$$B^{\hat{\phi}} = 2\Omega_F r B^{\hat{\theta}}, \tag{47}$$

to get the accretion rate expressed in terms of the magnetic field component perpendicular to the disk,

$$\dot{M}_+ = 4[B^{\hat{\theta}}(r_0) r_0]^2. \tag{48}$$

Then the total power for a disk with Poynting flux extended to $r_{out} = \infty$ is given by

$$P_{disk} = 4\{1 - [-u_0(r_{in})]\}[B^{\hat{\theta}}(r_0) r_0]^2, \tag{49}$$

where we take $-u_0(\infty) = 1$.

VI. DISCUSSION

In this work, we discuss a toy model for the magnetically dominated thin accretion disk. Assuming a two-dimensional disk dominated by Poynting flux where the viscous stress and the radiative transfers are ignored, the accretion flow in two dimensions is discussed in the background of the Kerr geometry. We have demonstrated that stationary accretion is possible with a corotating magnetic field with the same Keplerian angular velocity as that of matter in the disk in a stable orbit. Also, it is observed that the solution proposed by Blandford [2] is consistent with our result in the nonrelativistic limit (or at a large distance r from the center with sufficiently small a/r and a/M). It is shown in this toy model that the total Poynting power out of the disk depends on the accretion rate and the energy at the innermost stable orbit but not on the details of the electromagnetic field configuration.

Related issues to be discussed in the future are the possible solutions of the magnetic field configuration (not only the poloidal component of the magnetic field discussed by Ghosh [5] but also the toroidal component) analogous to [2] in the nonrelativistic limit and the effect of the magnetic field on the energy $(-u_0)$ and the angular momentum (u_ϕ) of a particle in a stable orbit, which has not been taken into account in this work.

We have thus far discussed only one specific aspect, the Poynting flux, of the accretion disk. For this two-dimensional disk model to be realistic and viable, there are many obstacles to overcome. For example, the viscous stress and radiation transfers should be included, which naturally leads to questions about the steady state [4] and the thin disk approximation assumed in this work. Moreover, recent work [22–24] on the accretion flow toward a black hole as a central object and on the magnetic coupling [14] between the disk and the black hole seems to indicate that the physics is more complex than the simplified two-dimensional model discussed in this work, particularly near the inner region of the disk.

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APPENDIX A: KERR GEOMETRY

In this appendix, the useful identities expressed in terms of explicit orthonormal components defined by ZAMO [19] are listed. Using the Boyer-Lindquist coordinates [25] in the natural units $G = c = 1$, the metric for the Kerr geometry [26] $g_{\mu\nu}$ is given by

$$(g_{\mu\nu}) = \begin{pmatrix} -(\alpha^2 - \varpi^2\beta^2) & 0 & 0 & \varpi^2\beta \\ 0 & \dfrac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ \varpi^2\beta & 0 & 0 & \varpi^2 \end{pmatrix},$$

where

$$\alpha = \frac{\rho\sqrt{\Delta}}{\Sigma}, \quad \beta = \frac{g_{0\phi}}{g_{\phi\phi}}, \quad \tilde{\omega} = \frac{\Sigma}{\rho}\sin\theta, \tag{A1}$$

$$\Delta = r^2 + a^2 - 2Mr, \quad \rho^2 = r^2 + a^2\cos^2\theta,$$

$$\Sigma^2 = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta. \tag{A2}$$

ZAMO's four velocity (a timelike unit vector orthogonal to the t-constant surface: $dx^\alpha U_\alpha = 0$) is given by

$$U_\mu = -\alpha(1,0,0,0). \tag{A3}$$

The electromagnetic field tensor can be expressed by the electric and magnetic fields as given by

$$F^\theta{}_\phi = \frac{\varpi}{\rho} B^{\hat{r}}, \quad F^\theta_r = -\frac{1}{\sqrt{\Delta}} B^{\hat{\phi}},$$

$$F^\theta{}_0 = \frac{1}{\rho^2}(\alpha\rho E^{\hat{\theta}} + \beta\rho\varpi B^{\hat{r}}), \tag{A4}$$

$$F^{0r} = \frac{\sqrt{\Delta}}{\alpha\rho} E^{\hat{r}}, \quad F^{0\phi} = \frac{1}{\alpha\varpi} E^{\hat{\phi}},$$

$$F^{\phi r} = -\frac{\beta\sqrt{\Delta}}{\alpha\rho} E^{\hat{r}} + \frac{\sqrt{\Delta}}{\varpi\rho} B^{\hat{\theta}}. \tag{A5}$$

APPENDIX B: SURFACE CURRENT ON A TWO-DIMENSIONAL ACCRETION DISK

Consider a conserved current $\mathcal{J}^\mu$ defined by

$$\mathcal{J}^\mu = J^\mu Y(\theta - \pi/2) + j^\mu, \tag{B1}$$

where the conserved current J^μ is the bulk current density satisfying the Maxwell equation,

$$F^{\mu\nu}{}_{;\nu} = 4\pi J^\mu, \quad J^\mu_{;\mu} = 0, \tag{B2}$$

and $Y(\theta - \pi/2)$ satisfies

$$Y(\theta - \pi/2)_{,\mu} = -\delta_{\theta\mu}\delta(\theta - \pi/2). \tag{B3}$$

From the conservation of the current,

$$\mathcal{J}^{\mu}{}_{;\mu}=0, \tag{B4}$$

we get

$$j^{\mu}{}_{;\mu}=\frac{1}{4\pi}F^{\theta\mu}_{;\mu}\delta(\theta-\pi/2). \tag{B5}$$

Then we can identify

$$j^{\mu}=\frac{1}{4\pi}F^{\theta\mu}\delta(\theta-\pi/2), \tag{B6}$$

since $F^{\theta\theta}$ vanishes identically. The analogous expression on the horizon can be found in [20].

Now the charge density defined by

$$\tilde{\rho}_e=\alpha j^0=\alpha\frac{F^{\theta 0}}{4\pi}\delta(\theta-\pi/2) \tag{B7}$$

can be rewritten as the surface charge density σ_e:

$$\int \tilde{\rho}_e\, dV\equiv\int \sigma_e\frac{\rho}{\sqrt{\Delta}}\varpi\, dr\, d\phi. \tag{B8}$$

Then we get the surface charge density in terms of the electric field given in Appendix A:

$$\sigma_e=-\frac{E^{\hat{\theta}}}{4\pi}. \tag{B9}$$

It is Gauss' law on the disk surface:

$$E^{\hat{\theta}}=-4\pi\sigma_e. \tag{B10}$$

Similarly, using the current density defined by

$$\tilde{j}^i=j^i-j^0(0,0,-\beta), \quad i=r,\theta,\phi, \tag{B11}$$

we get

$$\tilde{j}^r=-\frac{1}{4\pi}\frac{B^{\hat{\phi}}\sqrt{\Delta}}{\rho^2}\delta(\theta-\pi/2), \quad \tilde{j}^{\phi}=\frac{1}{4\pi}\frac{B^{\hat{r}}}{\rho\varpi}\delta(\theta-\pi/2), \tag{B12}$$

where $j^{\theta}=0$ by construction. The radial surface current density $K^{\hat{r}}$ and the surface current density in the ϕ direction $K^{\hat{\phi}}$ on the disk can be defined by

$$\int \alpha\tilde{j}^r\, d\Sigma_r=-\frac{1}{4\pi}\int \alpha B^{\hat{\phi}}\varpi\, d\phi\equiv\int \alpha K^{\hat{r}}\varpi\, d\phi, \tag{B13}$$

$$\int \alpha\tilde{j}^{\phi}\, d\Sigma_{\phi}=\frac{1}{4\pi}\int \alpha B^{\hat{r}}\frac{\rho}{\sqrt{\Delta}}dr\equiv\int \alpha K^{\hat{\phi}}\frac{\rho}{\sqrt{\Delta}}dr, \tag{B14}$$

to get

$$K^{\hat{r}}=-\frac{1}{4\pi}B^{\hat{\phi}}, \quad K^{\hat{\phi}}=\frac{1}{4\pi}B^{\hat{r}}, \tag{B15}$$

where $d\Sigma_i$ is the corresponding surface element.

This result can be summarized by Ampère's law on the disk surface:

$$\vec{B}=-4\pi\vec{K}\times\hat{\theta}. \tag{B16}$$

[1] R.V.E. Lovelace, Nature (London) **262**, 649 (1976).
[2] R.D. Blandford, Mon. Not. R. Astron. Soc. **176**, 465 (1976).
[3] M.C. Begelman, R.D. Blandford, and M.J. Rees, Rev. Mod. Phys. **56**, 255 (1984).
[4] G.V. Ustyugova, R.V.E. Lovelace, M.M. Romanova, H. Li, and S.A. Colgate, Astrophys. J. Lett. **541**, L21 (2000).
[5] P. Ghosh, Mon. Not. R. Astron. Soc. **315**, 89 (2000).
[6] P. Ghosh and M.A. Abramowicz, Mon. Not. R. Astron. Soc. **292**, 887 (1997).
[7] M. Livio, G.I. Ogilvie, and J.E. Pringle, Astrophys. J. **512**, 100 (1999).
[8] C. Fendt, Astron. Astrophys. **319**, 1025 (1997).
[9] H.K. Lee, R.A.M.J. Wijers, and G.E. Brown, Phys. Rep. **325**, 83 (2000).
[10] H.K. Lee, G.E. Brown, and R.A.M.J. Wijers, Astrophys. J. **536**, 416 (2000).
[11] L.-X. Li, Phys. Rev. D **61**, 084016 (2000).
[12] T. Piran, Phys. Rep. **314**, 575 (1998).
[13] H.K. Lee and H.K. Kim, J. Korean Phys. Soc. **36**, 188 (2000).
[14] L.-X. Li and B. Paczynski, Astrophys. J. Lett. **534**, L197 (2000).
[15] J.-P. Lasota, in *Theory of Accretion Disks 2*, edited by W.J. Duschl *et al.* (Kluwer, Dordrecht, 1994).
[16] M.A. Abramowicz, X.-M. Chen, M. Granath, and J.-P. Lasota, Astrophys. J. **471**, 762 (1996).
[17] C.F. Gammie and R. Popham, Astrophys. J. **498**, 313 (1998).
[18] R.D. Blandford and Z.L. Znajek, Mon. Not. R. Astron. Soc. **179**, 433 (1977).
[19] K.S. Thorne, R.H. Price, and D.A. Macdonald, *Black Holes: The Membrane Paradigm* (Yale University Press, New Haven, 1986).
[20] T. Damour, Phys. Rev. D **18**, 3598 (1978).
[21] S. Shapiro and S.A. Teukolsky, *Black Holes, White Dwarfs, and Neutron Stars* (John Wiley & Sons, New York, 1983).
[22] C.F. Gammie, Astrophys. J. Lett. **522**, L57 (1999).
[23] J.H. Krolik, Astrophys. J. Lett. **515**, L73 (1999).
[24] B. Punsly, Astrophys. J. **506**, 790 (1998).
[25] R.H. Boyer and R.W. Lindquist, J. Math. Phys. **8**, 265 (1967).
[26] R.P. Kerr, Phys. Rev. Lett. **11**, 237 (1963).